side, whence one obtains (assuming n particles inside Y)

$$\sum_{i=1}^{n} \mathbf{F}_i = \frac{d}{dt}\left[\sum_{i=1}^{n} (m_i\mathbf{v}_i)\right] = \frac{d\mathbf{G}}{dt} \tag{6}$$

That this is equivalent to Eq. (3) can be seen immediately by noting that

$$\frac{d}{dt}\left[\sum_{i=1}^{n} (m_i\mathbf{v}_i)\right] = \sum_{i=1}^{n} \frac{d}{dt}(m_i\mathbf{v}_i) + \frac{\partial \mathbf{G}}{\partial n}\frac{dn}{dt} \tag{7}$$

in which the first term on the right is the rate of change of momentum within Y and the second term is the rate at which momentum crosses the boundary. Note that the derivatives exist only in the formal or Dirac sense, since **G** is discontinuous. It can be shown, however, that

$$n = \sum_{i=1}^{\infty} A_i$$

and that $\partial\mathbf{G}/\partial n$ represent an average momentum of the particles crossing the boundary at a given instant.

Equation (6) proves that assumption 1 is incorrect and that the classical momentum equation is valid for variable mass systems. It is possible to obtain Eq. (6) directly from Eq. (3) without recourse to the division of space into Y and Y_E simply by proceeding from Eqs. (3–7) and thence to (6). It also is possible to obtain Eq. (6) by writing the momentum summation at time t and $t + \Delta t$ and taking the limit of the difference. This process yields Eq. (7) directly in terms of unit impulses on the boundary.

General Principle

Within a system, a particle located at (x_i, y_j, z_k) has a mass

$$m_{ijk}u(x - x_i)u(y - y_j)u(z - z_k)$$

The unit step functions are all open on the right. Thus the mass within Y becomes

$$m = \sum_{i,j,k} m_{ijk}u(x - x_i)u(y - y_j)u(z - z_k)$$

It is well known that the Stieltjes-Lebesgue integral with respect to a unit step is[2]

$$f(\xi) = \int f(x)\, d[u(x - \xi)]$$

and therefore

$$\mathbf{G} = \Sigma_i m_i\mathbf{v}_i = \int \mathbf{v}\, dm \tag{8}$$

By a simple limit procedure m may be made continuous, from which it can be seen that Eq. (6) applies to continuous mass systems. That this statement is true can be demonstrated easily. Let m be continuous; then $dm/dY = \rho$ is the mass density and $dY = dxdydz$. Select a system of orthogonal coordinates h_1, h_2, h_3 such that $h_3 = p(t)$ represents the outer boundary of Y; $p(t)$ is a function of time, since the control volume Y is defined by the mass, and mass is moving on the boundary. Then

$$= \int_a^b\int_c^d\int_e^f \mathbf{v}\rho\, dxdydz = \int_0^p\int_0^q\int_0^r \mathbf{v}\rho\, Jdh_1dh_2dh_3$$

a, b, c, d, e, f, and p are functions of time [these are equivalent to the upper summation limit in Eq. (6)], and J is the Jacobian of the transformation. Applying the standard rule for differentiating under the integral sign,[5] one obtains

$$\frac{d\mathbf{G}}{dt} = \int_0^p\int_0^q\int_0^r \frac{\partial}{\partial t}(\rho\mathbf{v}J)dh_1dh_2dh_3 + \frac{dp}{dt}R(p) \tag{9}$$

where

$$R(p) = \int_0^q\int_0^r \mathbf{v}\rho\, Jdh_1dh_2$$

at $h_3 = p$. Equation (9) represents the rocket equation, since the first term on the right is the rate of change of momentum in Y, and the second term is the rate at which momentum crosses the boundary.

Conclusions

1) If the momentum of a mass system is represented by the Stieltjes-Lebesgue integral

$$\mathbf{G} = \int \mathbf{v}\, dm \tag{10}$$

in which the integral is taken over all the mass in the system, then Newton's equation

$$\Sigma\mathbf{F} = d\mathbf{G}/dt \tag{11}$$

is valid for all continuous and discontinuous mass system and for time-variable as well as time-fixed masses.

2) The equation

$$\sum_{i=1}^{n} \mathbf{F}_i = \frac{d}{dt}\left(\sum_{i=1}^{n} m_i\mathbf{v}_i\right) \tag{12}$$

is valid for time variable mass systems.

3) In Eq. (12)

$$n = \sum_{i=1}^{\infty} A_i(T_1, T_2, t) \tag{13}$$

References

[1] Meriam, J. L., "Momentum equation for variable mass," Aerospace Eng. 21, 52–53 (July 1962).

[2] Caratheodory, C., *Vorlesungen Uber Reelle Funktionen* (Chelsea Publishing Co., New York, 1948).

[3] Goldstein, H., *Classical Mechanics* (Addison-Wesley Press, Inc., Cambridge, Mass., 1950), pp. 4–10.

[4] Cramer, H., *Mathematical Methods of Statistics* (Princeton University Press, Princeton, N. J., 1946), p. 53 ff.

[5] Hildebrand, F. B., *Advanced Calculus for Engineers* (Prentice-Hall Inc., New York, 1949), 1st ed., pp. 352–353.

Singular Line of the Method of Integral Relations

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In the method of integral relations, as applied by Bielotserkovskii to the blunt-body problem,[1–3] there occurs a singular line that cuts across the shock layer. The equation of the singular line is obtained by equating the determinant of the system of resulting ordinary differential equations to zero. This equation, however, gives no information about the position of the singular line, and its physical significance has not been explained in the literature.

Bielotserkovskii's method has been criticized often for its reliance on the boundary conditions given on the singular line. The main objection to such a procedure seemed to be the fact that coordinates of the singular line depend not only on the solution but also on the choice of the coordinate system. This note explains this apparent discrepancy by giving the singular line a physical interpretation.

A problem of a blunt body placed at zero incidence in a supersonic flow is considered. Perfect gas is assumed. Without loss of generality, the s,n coordinate system is chosen,

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where s is an abscissa measured along the body and n is an ordinate measured away from the body along the normal. The singular line then is given by[1–5]

$$v^2 = [(\gamma - 1)/(\gamma + 1)](1 - u^2) \quad (1)$$

where v and u are components of velocity in the s and n direction, respectively, referred to the limiting speed. The choice of the s,n system renders Eq. (1) free from explicit dependence on the shape of the body.

As the partial differential equations governing the flow are integrated in the n direction across the shock layer, there results an approximating system of ordinary differential equations in a single variable s; this system then is integrated numerically. The ordinary derivatives of the unknowns at any value of s are expressed in terms of the data along the ordinate s = const. The ordinate, therefore, plays the role of a data line for the continuation of the solution in direction of increasing s.

The significance of relation (1) can be revealed as follows. The local speed of sound, nondimensionalized with respect to the limiting speed, is

$$a = \{[(\gamma - 1)/2](1 - u^2 - v^2)\}^{1/2}$$

Equation (1) may be written as

$$\{1 + [(\gamma - 1)/2]\}v^2 = [(\gamma - 1)/2](1 - u^2)$$

$$v^2 = [(\gamma - 1)/2](1 - u^2 - v^2) = a^2$$

Therefore, along the singular line the s component of velocity attains sonic value. Taking square roots and dividing by the magnitude of velocity q, one obtains

$$v/q = \pm a/q = \pm 1/M = \sin(\pm\mu) \quad (2)$$

where μ is the Mach angle. Equation (2) implies that the data line, s = const, becomes tangent to a characteristic at every point of the singular line along which Eq. (1) holds. In other words, the singular line is a locus of points at which the direction of integration becomes normal to a characteristic. Obviously, if the coordinate system is changed, so is the direction of integration, and the singularity moves to a different location. Incidentally, in spherical coordinates (r,θ), Eq. (1) has the same form if u and v are taken to be velocity components in r and θ directions.

Using the present interpretation, it is now possible to explain the well-known fact that the point of intersection of the singular line with the shock wave is nonsingular. This is so because the angle between the shock and the velocity immediately behind the shock is less than Mach angle. Consequently, the shock layer downstream of the singular line lies outside the range of influence of the point of intersection. This is borne out by the fact that the equation for the derivative of the shock angle, as given by integral method, is nonsingular.[4,5]

The implications of identifying the singular line of the integral method with the locus of tangency of data lines with characteristics are obvious in the light of the theory of characteristics. However, it should be pointed out that the occurrence of a singular line of the type described here is not a peculiarity of the method of integral relations. The method introduces no artificiality in the form of a singular line; on the contrary, it is rather remarkable to observe a complete agerement between an approximate method such as Bielotserkovskii's and the theory of characteristics.

References

[1] Bielotserkovskii, O. M., "Flow past a symmetric profile with detached shock wave," Prikl. Mat. i Mekh. (Appl. Math. Mech.) 22 (1958).

[2] Bielotserkovskii, O. M., "On the calculation of flow past axisymmetric bodies with detached shock waves using an electronic computing machine," Prikl. Mat. i. Mekh. (Appl. Math. Mech.) **24,** 511–517 (1960).

[3] Bielotserkovskii, O. M., "Calculation of flow past axisymmetric bodies with detached shock waves," *Equations for Calculations and Tables of Flow Fields* (Vichislitielnii Centr Akademii Nauk SSSR, Moscow, 1961).

[4] Traugott, S. C., "An approximate solution of the supersonic blunt body problem for prescribed arbitrary axisymmetric shapes," Research Rept. RR-13, Martin Co., Baltimore, Md. (August 29, 1858).

[5] Gold, R. and Holt, M., "Calculation of supersonic flow past a flat-head cylinder by Bielotserkovskii's method," Brown Univ., Div. Appl. Math., AFOSR TN 59-199 (March 1959).

Simultaneous Gas-Phase and Surface Atom Recombination for Stagnation Boundary Layer

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Nomenclature

C = $\rho\mu$ ratio defined by Eq. (3)
c = total atom mass fraction
c_2 = atom mass fraction of nitrogen atoms
D = binary diffusion coefficient
h_f = frozen total enthalpy
Δh^0 = heat of recombination
K_g = equivalent surface reaction constant for gas-phase reaction
$\hat{K}_g$ = parameter defined by Eq. (5)
K_w = specific catalytic surface recombination constant
k_0 = constant portion of recombination coefficient, 1.56×10^{20}
m = c/c_e
p = pressure, atm
q = heat transfer to surface
R = universal gas constant, 82.06 cm³-atm/mole-°K
r = distance from the axis of symmetry to the surface
s = function defined by Eq. (3)
Sc = Schmidt number, $\mu/\rho D$
T = absolute temperature, °K
u = streamwise velocity
x = streamwise distance
y = distance normal to surface
β = $(du_e/dx)_0$, sec for Eq. (8)
Γ_g = Damkhöler number for gas-phase recombination
Γ_w = Damkhöler number for surface recombination
ϵ = {0 for two-dimensional body; 1 for axisymmetric body}
η = function defined by Eq. (3)
μ = viscosity
ν = kinematic viscosity, μ/ρ
ρ = density

Subscripts

E = equilibrium
e = edge of boundary layer
f = frozen
0 = stagnation point
w = wall
∞ = freestream

Introduction

The chemical state of nonequilibrium boundary layers about hypersonic vehicles is of considerable interest. The

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